Nonlinear Response from Transport Theory and Quantum Field Theory at Finite Temperature

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Abstract

We study nonlinear response in weakly coupled hot ϕ^4 theory. We obtain an expression for a quadratic shear viscous response coefficient using two different formalisms: transport theory and response theory. The transport theory calculation is done by assuming a local equilibrium form for the distribution function and expanding in the gradient of the local four dimensional velocity field. By doing a gradient expansion on the Boltzmann equation we obtain a hierarchy of equations for the coefficients of this expansion.

To do the response theory calculation we use Zubrave's techniques in nonequilibrium statistical mechanics to derive a generalized Kubo formula. Using this formula allows us to obtain the quadratic shear viscous response from the three-point retarded green function of the viscous shear stress tensor. We use the closed time path formalism of real time finite temperature field theory to show that this three-point function can be calculated by writing it as an integral equation involving a four-point vertex. This four-point vertex can in turn be obtained from an integral equation which represents the resummation of an infinite series of ladder and extended-ladder diagrams.

The connection between transport theory and response theory is made when we show that the integral equation for this four-point vertex has exactly the same form as the equation obtained from the Boltzmann equation for the coefficient of the quadratic term of the gradient expansion of the distribution function. We conclude that calculating the quadratic shear viscous response using transport theory and keeping terms that are quadratic in the gradient of the velocity field in the expansion of the Boltzmann equation is equivalent to calculating the quadratic shear viscous response from response theory using the next-to-linear response Kubo formula, with a vertex given by an infinite resummation of ladder and extended-ladder diagrams.

I. INTRODUCTION

Fluctuations occur in a system perturbed slightly away from equilibrium. The responses to these fluctuations are described by transport coefficients which characterize the dynamics of long wavelength, low frequency fluctuations in the medium [1,2]. The investigation of transport coefficients in high temperature gauge theories is important in cosmological applications such as electroweak baryogenesis [3] and in the context of heavy ion collisions [4].

There are two basic methods to calculate transport coefficients: transport theory and response theory [5–12]. Using the transport theory method one starts from a local equilibrium form for the distribution function and performs an expansion in the gradient of the four-velocity field. The coefficients of this expansion are determined from the classical Boltzmann equation [10,12]. In the response theory approach one divides the Hamiltonian into a bare piece and a perturbative piece that is linear in the gradient of the four-velocity field. One uses standard perturbation theory to obtain the Kubo formula for the viscosity in terms of retarded green functions [6]. These green functions are then evaluated using equilibrium quantum field theory. As is typical in finite temperature field theory, it is not sufficient to calculate perturbatively in the coupling constant: there are certain infinite sets of diagrams that contribute at the same order in perturbation theory and have to be resummed [13,14].

In this paper, we want to compare these two methods. The response theory approach allows us to calculate transport coefficients from first principles using the well understood methods of quantum field theory. On the other hand, the transport theory approach involves the use of the Boltzmann equation which is itself derived from some more fundamental theory using, among other things, the quasiparticle approximation. In this sense, the response theory approach is more fundamental than the transport theory method. However, the response theory approach can be quite difficult to implement, even for a high temperature weakly coupled scalar theory, because of the need to resum infinite sets of diagrams [13,14]. These considerations motivate us to understand more precisely the connection between the more practical transport theory method, and the more fundamental response theory approach.

Some progress has already been made in this direction. It has been shown that keeping only terms which are linear in the gradient expansion in the transport theory calculation is equivalent to using the linear response approximation to obtain the usual Kubo formula for the shear viscosity in terms of a retarded two-point function, and calculating that two-point function using standard equilibrium quantum field theory techniques to resum an infinite set of ladder diagrams [10,15,16]. This result is not surprising since it has been known for some time that ladder diagrams give large contributions to n-point functions with ultra soft external lines [14].

To date, calculations of transport coefficients have been limited to linear response. In some physical situations however nonlinear response can be important, especially for relativistic gauge theories [17–20]. It is therefore of interest to study nonlinear response. In this paper we study nonlinear response to fluctuations in weakly coupled high temperature scalar ϕ^4 theory using both the transport method and the response theory approach. We study the relationship between these two approaches at the level of quadratic response.

This paper will be organized as follows. In section II we define shear viscosity and quadratic shear viscous response using a hydrodynamic expansion of the energy-momentum tensor which includes up to quadratic terms in the gradient of the four-velocity. In section III we calculate viscosity using the transport theory method by performing a gradient expansion on the Boltzmann equation and obtaining a hierarchy of equations. In section IV we derive the quadratic response Kubo formula by using Zubrave's techniques in nonequilibrium statistical mechanics. We obtain an expression which relates the quadratic shear viscous response coefficient to the three-point retarded green function of the viscous shear stress tensor. Starting from this generalized Kubo formula we calculate the quadratic shear viscous response using standard techniques of finite temperature quantum field theory. We show that the quadratic shear viscous response can be obtained as an integral over a four-point vertex. This four-point vertex satisfies an integral equation involving terms which are quadratic in a retarded three-point function, which itself satisfies a linear integral equation. These two integral equations represent the resummation of an infinite series of ladder and extended-ladder diagrams. We show that these integral equations, which represent ladder and extended-ladder resummations, have the same form as the first two equations in the hierarchy obtained from expanding the Boltzmann equation. We discuss our results and present our conclusions section V.

II. VISCOSITY

In a system that is out of equilibrium, the existence of gradients in thermodynamic parameters like the temperature and the four dimensional velocity field give rise to thermodynamic forces. These thermodynamic forces lead to deviations from the equilibrium expectation value of the energy momentum tensor which are characterized by transport coefficients like the thermal conductivity and the shear and bulk viscosities. In order to separate these different physical processes we decompose the energy-momentum tensor as,

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - p\Delta^{\mu\nu} + P^\mu u^\nu + P^\nu u^\mu + \pi^{\mu\nu}\,; \qquad \Delta_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu. \tag{2.1}$$

The quantities ϵ, p, P_μ and $\pi_{\mu\nu}$ have the physical meanings of internal energy density, pressure, heat current and viscous shear stress, respectively. The four vector $u_\mu(x)$ is the four dimensional four-velocity field which satisfies $u^\mu(x)u_\mu(x) = 1$. The expansion coefficients are given by

$$\begin{aligned}
\epsilon &= u_\alpha u_\beta T^{\alpha\beta} \\
p &= -\frac{1}{3}\Delta_{\alpha\beta}T^{\alpha\beta} \\
P_\mu &= \Delta_{\mu\alpha}u_\beta T^{\alpha\beta} \\
\pi_{\mu\nu} &= (\Delta_{\mu\alpha}\Delta_{\nu\beta} - \frac{1}{3}\Delta_{\nu\mu}\Delta_{\alpha\beta})T^{\alpha\beta}.
\end{aligned} \tag{2.2}$$

The viscosity is obtained from the expectation value of the viscous shear stress part of the energy momentum tensor. We expand in gradients of the four-velocity field and write,

$$\begin{aligned}
&\delta\langle\pi_{\mu\nu}\rangle = \eta^{(1)}H_{\mu\nu} + \eta^{(2)}H^{T2}_{\mu\nu} + \cdots \qquad (2.3)\\
&H_{\mu\nu} = \partial_\mu u_\nu + \partial_\nu u_\nu - \frac{2}{3}\Delta_{\mu\nu}\Delta_{\rho\sigma}\partial^\rho u^\sigma \\
&H^{T2}_{\mu\nu} := H_{\mu\rho}H^\rho{}_\nu - \frac{1}{3}\Delta_{\mu\nu}H_{\rho\sigma}H^{\rho\sigma}
\end{aligned}$$

where $\eta^{(1)}$ and $\eta^{(2)}$ are the coefficients of the terms that are linear and quadratic respectively in the gradient of the four-velocity. The first coefficient is the usual shear viscosity. The second has has not been widely discussed in the literature – we shall call it the quadratic shear viscous response.

Throughout this paper we work with ϕ^4 theory. The Lagrangian for this theory is

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu \phi)^2 - m^2\phi^2] - \frac{\lambda}{4!}\phi^4 \tag{2.4}$$

and the coupling constant is assumed to be small: $\lambda \ll 1$.

III. VISCOSITY FROM TRANSPORT THEORY

Kinetic theory and the Boltzmann equation can be used to calculate transport properties of dilute many-body systems. One assumes that, except during brief collisions, the system can be considered as being composed of classical particles with well defined position, energy and momentum. This picture is valid when the mean free path is large compared with the Compton wavelength of the particles. At high temperature the typical mean free path of thermal excitations is $\mathcal{O}(1/\lambda^2 T)$ and is always larger than the typical Compton wavelength of effective thermal oscillations which is $\mathcal{O}(1/\sqrt{\lambda}T)$ [10]. We introduce a phase space distribution function $f(x, \underline{k})$ which describes the evolution of the phase space probability density for the fundamental particles comprising a fluid. In this expression and in the following equations the underlined momenta are on shell, since we are describing a system of particles. The form for $f(x, \underline{k})$ in local equilibrium is,

$$f^{(0)} = \frac{1}{e^{\beta(x)u_\mu(x)\underline{k}^\mu} - 1} := n_k\,; \quad N_k := 1 + 2n_k\,. \tag{3.1}$$

We study the Boltzmann equation in the hydrodynamic regime where we consider times which are long compared to the mean free time and describe the relaxation of the system in terms of long wavelength fluctuations in locally conserved quantities. For a simple fluid without any additional conserved charges, the only locally conserved quantities are energy and momentum. To solve the Boltzmann equation in this near equilibrium hydrodynamic regime, we expand the distribution function around the local equilibrium form using a gradient expansion. We go to a local rest frame in which we can write $\vec{u}(x) = 0$. Note that this does not imply that gradients of the form $\partial_i u_j$ must be zero. In the local rest frame (2.3) becomes,

$$\begin{aligned}
\delta\langle\pi_{ij}\rangle &= -\eta^{(1)} H_{ij} + \eta^{(2)} H^{T2}_{ij} + \cdots \\
H_{ij} &= \partial_i u_j + \partial_j u_i - \frac{2}{3}\delta_{ij}(\vec{\partial}\cdot\vec{u}) \\
H^{T2}_{ij} &:= H_{ik}H^k{}_j - \frac{1}{3}\delta_{ij}\ H_{lm}H_{lm}
\end{aligned} \tag{3.2}$$

In all of the following expressions we keep only linear terms that contain one power of H_{ij} and quadratic terms that contain two powers of H_{ij}, since these are the only terms that contribute to the viscosity coefficients we are trying to calculate.

We write,

$$f = f^{(0)} + f^{(1)} + f^{(2)} + \cdots \tag{3.3}$$

with,

$$f^{(1)} \sim \underline{k}_\mu \partial^\mu f^{(0)}\,; \qquad f^{(2)} \sim \underline{k}_\mu \partial^\mu f^{(1)}. \tag{3.4}$$

Using (3.1) we obtain,

$$\begin{aligned}
\underline{k}_\mu \partial^\mu f^{(0)} &= -\beta\, n_k(1+n_k)\,\frac{1}{2} I_{ij}(k) H_{ij} \qquad (3.5)\\
f^{(1)} &:= -n_k(1+n_k)\phi_k\,; \quad \phi_k = \beta\frac{1}{2} B_{ij}(\underline{k}) H_{ij}\\
\underline{k}_\mu \partial^\mu f^{(1)} &= \beta^2\, n_k(1+n_k) N_k\,\frac{1}{2} I_{ij}(k) H_{ij}\,\frac{1}{2} B_{lm}(\underline{k}) H_{lm}\\
f^{(2)} &:= n_k(1+n_k) N_k \theta_k\,; \quad \theta_k := \beta^2\frac{1}{4} C_{ijlm}(\underline{k}) H_{ij} H_{lm}
\end{aligned}$$

where we define

$$\hat{I}_{ij}(k) = (\hat{k}_i \hat{k}_j - \frac{1}{3}\delta_{ij})\,; \quad I_{ij}(k) = k^2 \hat{I}_{ij}(k) \tag{3.6}$$

and write,

$$B(\underline{k})_{ij} = \hat{I}_{lm}(k) B(\underline{k})\,; \qquad C_{ijlm}(\underline{k}) = \hat{I}_{ij}(k)\hat{I}_{lm}(k) C(\underline{k}). \tag{3.7}$$

The viscous shear stress part of the energy momentum tensor is given by

$$\langle \pi_{ij} \rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k} f\,(k_i k_j - \frac{1}{3}\delta_{ij} k^2)\,. \tag{3.8}$$

Using the expansion (3.3) we get,

$$\langle \pi_{ij} \rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k} [f^{(0)} + f^{(1)} + f^{(2)}]\,(k_i k_j - \frac{1}{3}\delta_{ij} k^2)\,. \tag{3.9}$$

The lowest order term is identically zero. We obtain the linear and quadratic contributions by substituting in (3.5) and (3.7). We use $H_{ii} := 0$ and the following results which are obtained from rotational invariance:

$$\begin{aligned}
k_i k_j B(\underline{k}) &\rightarrow \frac{1}{3}\delta_{ij} k^2 B(\underline{k})\\
k_i k_j \hat{k}_l \hat{k}_m B(\underline{k}) &\rightarrow \frac{1}{15}(\delta_{ij}\delta_{lm} + \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}) k^2 B(\underline{k})\\
k_i k_j \hat{k}_l \hat{k}_m \hat{k}_a \hat{k}_b C(\underline{k}) &\rightarrow \frac{1}{105}[\,\delta_{ab}(\delta_{lm}\delta_{ij} + \delta_{lj}\delta_{mi} + \delta_{li}\delta_{mj}) + \delta_{al}(\delta_{bm}\delta_{ij} + \delta_{bi}\delta_{mj} + \delta_{bj}\delta_{mi})\\
&\quad + \delta_{am}(\delta_{bl}\delta_{ij} + \delta_{bj}\delta_{li} + \delta_{bi}\delta_{lj}) + \delta_{ai}(\delta_{bl}\delta_{mj} + \delta_{bm}\delta_{lj} + \delta_{bj}\delta_{ml})\\
&\quad + \delta_{aj}(\delta_{bl}\delta_{mi} + \delta_{bm}\delta_{li} + \delta_{bi}\delta_{ml})\,]\, k^2 C(\underline{k}) \qquad (3.10)
\end{aligned}$$

We obtain,

$$\delta\langle\pi_{ij}\rangle = -\frac{\beta}{15}\int\frac{d^3k}{(2\pi)^3 2\omega_k}n_k(1+n_k)k^2B(\underline{k})\,H_{ij} + \frac{2\beta^2}{105}\int\frac{d^3k}{(2\pi)^3 2\omega_k}[n_k(1+n_k)N_k]k^2C(\underline{k})H^{T2}_{ij}$$

Comparing with (3.2) we have,

$$\eta^{(1)} = \frac{\beta}{15}\int\frac{d^3k}{(2\pi)^3 2\omega_k}n_k(1+n_k)k^2B(\underline{k}) \tag{3.11}$$

$$\eta^{(2)} = \frac{2\beta^2}{105}\int\frac{d^3k}{(2\pi)^3 2\omega_k}[n_k(1+n_k)N_k]k^2C(\underline{k}) \tag{3.12}$$

Thus we have shown that the shear viscosity and the quadratic shear viscous response can be obtained from the functions $B(\underline{k})$ and $C(\underline{k})$ respectively. These two functions are the coefficients of the linear and quadratic terms in the gradient expansion of the distribution function. In the next section we will show that these functions can be obtained from the first two equations in the hierarchy of equations obtained from the gradient expansion of the Boltzmann equation.

A. Expansion of the Boltzmann Equation

The Boltzmann equation describes the evolution of the distribution function $f(x,\underline{k})$ and can be used to obtain integral equations for the functions $B(\underline{k})$ and $C(\underline{k})$ defined in (3.5) and (3.7). The Boltzmann equation has the form:

$$\underline{k}_\mu\partial^\mu f(x,\underline{k}) = \mathcal{C}[f] \tag{3.13}$$

where $\mathcal{C}[f]$ is the collision term:

$$\mathcal{C}[f] = \frac{1}{2}\int_{123} d\,\Gamma_{12\leftrightarrow 3k}[f_1f_2(1+f_3)(1+f_k) - (1+f_1)(1+f_2)f_3f_k] \tag{3.14}$$

with $f_i := f(x,\underline{p}_i)$, $f_k := f(x,\underline{k})$. The symbol $d\,\Gamma_{12\leftrightarrow 3k}$ represents the differential transition rate for particles of momentum P_1 and P_2 to scatter into momenta P_3 and K and is given by

$$d\,\Gamma_{12\leftrightarrow 3k} := \frac{1}{2\omega_k}|\mathcal{T}(\underline{k},\underline{p}_3,\underline{p}_2,\underline{p}_1|^2\Pi^3_{i=1}\frac{d^3p_i}{(2\pi)^3 2\omega_{p_i}}(2\pi)^4\delta(\underline{p}_1+\underline{p}_2-\underline{p}_3-\underline{k}) \tag{3.15}$$

where $\mathcal{T}$ is the multiparticle scattering amplitude. Using the expansion (3.3) and (3.5) produces a hierarchy of equations.

1. *First Order Boltzmann Equation*

The first order equation is:

$$\underline{k}^\mu\partial_\mu f^0(x,\underline{k}) = \mathcal{C}[f^{(0)};f^{(1)}] \tag{3.16}$$

where we keep terms linear in $f^{(1)}$ on the right hand side. The left hand side gives,

$$\underline{k}^\mu \partial_\mu f^0(x,\underline{k}) = -n_k(1+n_k)\beta\,\frac{1}{2}I_{ij}(k)H_{ij} \tag{3.17}$$

The right hand side is

$$\mathcal{C}[f^{(1)}] = \frac{1}{2}\int_{123} d\,\Gamma_{12\leftrightarrow 3k}(1+n_1)(1+n_2)n_3n_k(\phi_k+\phi_3-\phi_1-\phi_2) \tag{3.18}$$

Using the definition of ϕ given in (3.5) and comparing the coefficients of H_{ij} on both sides of (3.16) we obtain,

$$I_{ij}(k) = \frac{1}{2}\int_{123} d\,\Gamma_{12\leftrightarrow 3k}\frac{(1+n_1)(1+n_2)n_3}{1+n_k}[B_{ij}(\underline{p}_1)+B_{ij}(\underline{p}_2)-B_{ij}(\underline{k})-B_{ij}(\underline{p}_3)\,] \tag{3.19}$$

This result is an inhomogeneous linear integral equation which can be solved self-consistently to obtain the function $B_{ij}(\underline{k})$.

2. Second Order Boltzmann Equation

The second order contribution to (3.13) is,

$$\underline{k}^\mu \partial_\mu f^{(1)}(x,\underline{k}) = \mathcal{C}[f^{(0)};f^{(1)};f^{(2)}] \tag{3.20}$$

where we keep terms linear in $f^{(2)}$ and quadratic in $f^{(1)}$ on the right hand side. Using (3.5) the left hand side becomes,

$$\underline{k}^\mu \partial_\mu f^{(1)} = n_k(1+n_k)N_k\beta^2\frac{1}{4}I_{ij}(k)B_{lm}(\underline{k})H_{ij}H_{lm}$$

and the right hand side gives,

$$\begin{aligned}C^{(2)}[f^{(0)};f^{(1)};f^{(2)}] = &\frac{1}{2}\int_{123} d\,\Gamma_{12\leftrightarrow 3k}(1+n_1)(1+n_2)n_3n_k\\ &[(N_1\theta_1+N_2\theta_2-N_3\theta_3-N_k\theta_k)\\ &+\frac{1}{2}((N_1+N_2)\phi_1\phi_2-(N_k+N_3)\phi_3\phi_k+(N_3-N_1)\phi_1\phi_3\\ &+(N_k-N_1)\phi_1\phi_k+(N_3-N_2)\phi_3\phi_2+(N_k-N_2)\phi_k\phi_2)]\end{aligned} \tag{3.21}$$

Using the definitions of ϕ and θ given in (3.5) and comparing the coefficients of $H_{ij}H_{lm}$ on both sides we obtain,

$$\begin{aligned}n_k(1+n_k)N_kI_{ij}(k)B_{lm}(\underline{k}) = &\frac{1}{2}\int_{123} d\,\Gamma_{12\leftrightarrow 3k}(1+n_1)(1+n_2)n_3n_k\\ &\{[N_1C_{ijlm}(\underline{p}_1)+N_2C_{ijlm}(\underline{p}_2)-N_kC_{ijlm}(\underline{k})-N_3C_{ijlm}(\underline{p}_3)]\\ &+\frac{1}{2}[((N_1+N_2)B_{ij}(\underline{p}_1)B_{lm}(\underline{p}_2)-(N_k+N_3)B_{ij}(\underline{p}_3)B_{lm}(\underline{k})+(N_3-N_1)B_{ij}(\underline{p}_1)B_{lm}(\underline{p}_3)\\ &+(N_k-N_1)B_{ij}(\underline{p}_1)B_{lm}(\underline{k})+(N_3-N_2)B_{ij}(\underline{p}_3)B_{lm}(\underline{p}_2)+(N_k-N_2)B_{ij}(\underline{k})B_{lm}(\underline{p}_2)]\}\end{aligned} \tag{3.22}$$

This integral equation can be solved self consistently for the quantity $C_{ijlm}(\underline{k})$ using the result for $B_{ij}(\underline{k})$ from (3.19).

IV. VISCOSITY FROM FIELD THEORY

The Kubo formulae allow us to use quantum field theory to calculate nonequilibrium transport coefficients. The results should be the same as those obtained in the previous section using transport theory. To do calculations in a system that is out of equilibrium, the Hamiltonian is separated into an equilibrium piece H_0 and a nonequilibrium piece H_{ext} which depends on the gradients of the thermodynamic parameters: the four-velocity field and the inverse temperature field. For systems close to equilibrium the nonequilibrium piece of the Hamiltonian can be treated as a perturbation and the deviations of physical quantities from their equilibrium values can be calculated perturbatively. The linear response calculation includes only the first order contribution to this expansion and gives transport coefficients that can be expressed as integrals of retarded two-point Green functions over space and time. One of the results we obtain in this paper is that the quadratic shear viscous response coefficient can be written in terms of a retarded three-point function. Throughout this section we use capital letters to denote four-vectors and small letters for three-vectors. We also define $\int d^4p/(2\pi)^4 := \int dP$.

To calculate the expectation value of an operator we take the trace over the density matrix. We follow the presentation of [8]. We work with the density matrix in the Heisenberg representation which satisfies,

$$\frac{\partial \rho}{\partial t} = 0 \tag{4.1}$$

and can be written as,

$$\rho = \frac{e^{-A+B}}{\mathrm{Tr} e^{-A+B}} \tag{4.2}$$

where

$$\begin{aligned} A &= \int d^3x F^\nu T_{0\nu}, \\ B &= \int d^3x \int_{-\infty}^{t} dt' e^{\epsilon(t'-t)} T_{\mu\nu}(x,t') \partial^\mu F^\nu(x,t') \end{aligned} \tag{4.3}$$

with $F^\mu = \beta u^\mu$ and ϵ to be taken to zero at the end. In this expression A is the equilibrium part of the Hamiltonian and B is a perturbative contribution that is linear in the gradient of the four-velocity field. Note that in the local rest frame $u^\mu = (1,0,0,0)$ and $[A]^{LRF} = \int d^3x\, \beta T_{00}$. Using the identity for the exponential function of two operators

$$e^{\beta(\hat{a}+\hat{b})} = e^{\beta\hat{a}}[1 + \int_0^\beta d\lambda e^{\lambda\hat{a}} \hat{b} e^{-\lambda(\hat{a}+\hat{b})}] \tag{4.4}$$

we can expand

$$\begin{aligned} e^{-A+B} = e^{-A}[1 &+ \int_0^1 d\lambda e^{\lambda A} B e^{-\lambda A} \\ &+ \int_0^1 d\lambda \int_0^\lambda d\tau e^{\lambda A} B e^{-\lambda A} e^{\tau A} B e^{-\tau A} + \mathcal{O}(B^3)] \end{aligned} \tag{4.5}$$

Thus the density matrix can be written as

$$\rho = \rho_0 \left[1 + \int_0^1 d\lambda (B(\lambda) - \langle B(\lambda) \rangle) + \int_0^1 d\lambda \int_0^\lambda d\tau (B(\lambda)B(\tau) - \langle B(\lambda)B(\tau) \rangle) \right.$$
$$\left. - \int_0^1 d\lambda \int_0^1 d\lambda' (\langle B(\lambda) \rangle B(\lambda') - \langle B(\lambda) \rangle \langle B(\lambda') \rangle \right] + O(B^3) \tag{4.6}$$

where

$$\rho_0 = \frac{e^{-A}}{\mathrm{Tr} e^{-A}} \tag{4.7}$$

is the local equilibrium density matrix.

A. Viscosity as an Expansion in Green Functions of Composite Operators

1. Linear Response

Using the first three terms of (4.6) produces the linear response approximation for the deviation of the expectation value of the viscous shear stress part of the energy momentum tensor from the equilibrium value:

$$\begin{aligned} \delta \langle \pi_{\mu\nu}(x,t) \rangle^l &= \langle \pi_{\mu\nu}(x,t) \rangle^l - \langle \pi_{\mu\nu}(x,t) \rangle_0 \\ &= \int d^3x' \int_{-\infty}^t dt' e^{\epsilon(t'-t)} (\pi_{\alpha\beta}(x',t'), \pi_{\mu\nu}(x,t)) \partial^\alpha F^\beta(x',t'') \end{aligned} \tag{4.8}$$

where the correlation function $(\pi_{\alpha\beta}(x',t'), \pi_{\mu\nu}(x,t))$ is defined as,

$$(\pi_{\mu\nu}(x,t), \pi_{\alpha\beta}(x',t')) = \frac{1}{\beta} \int_0^\beta d\tau \langle (\pi_{\alpha\beta}(x', t' + i\tau) \pi_{\mu\nu}(x,t) - \langle \pi_{\alpha\beta}(x', t' + i\tau) \rangle_0 \pi_{\mu\nu}(x,t) \rangle_0 \tag{4.9}$$

and $\langle \cdots \rangle_0 := \mathrm{Tr}(\rho_0 \cdots)$. From now on we drop the subscript 0 on the correlation functions. We take the initial time t_0 to minus infinity and we assume the system is in equilibrium at $t = t_0$ and that the external forces are switched on adiabatically. We assume that changes in the thermodynamic forces are small enough over the correlation length of the correlation functions that factors $F_\mu = \beta u_\mu$ can be taken out of the integral. Using (2.2) and rotational invariance we have,

$$\pi_{\mu\nu}\pi_{\alpha\beta} = \frac{1}{10}(\Delta_{\mu\alpha}\Delta_{\nu\beta} + \Delta_{\mu\beta}\Delta_{\nu\alpha} - \frac{2}{3}\Delta_{\mu\nu}\Delta_{\alpha\beta})\pi_{\sigma\tau}\,\pi^{\sigma\tau} \tag{4.10}$$

Using this result we obtain,

$$\begin{aligned} \delta \langle \pi_{\mu\nu} \rangle^l &= \frac{H_{\mu\nu}}{10} \int d^3x' \int_{-\infty}^t dt' e^{\epsilon(t'-t)} \\ &\int_0^\beta (\langle \langle (\pi(x', t+i\tau)\pi(x,t) \rangle - \langle \langle \pi(x', t+i\tau) \rangle \pi(x,t) \rangle \rangle) \end{aligned} \tag{4.11}$$

where we have written $\pi_{\sigma\tau}\pi^{\sigma\tau} := \pi\pi$. If we assume that correlations vanish as $t' \to -\infty$ this expression can be rewritten as,

$$\begin{aligned}\delta\langle\pi_{\mu\nu}\rangle^l =& \frac{H_{\mu\nu}}{10}\int d^3x'\int_{-\infty}^{t} dt' e^{\epsilon(t'-t)} \\ & \int_0^\beta \int_{-\infty}^{t'} dt'' \frac{d}{dt''}(\langle(\pi(x',t''+i\tau)\pi(x,t)\rangle - \langle\langle\pi(x',t''+i\tau)\rangle\pi(x,t)\rangle)\end{aligned} \tag{4.12}$$

Using $\frac{\partial}{dt''}f(t''+i\tau) = -i\frac{\partial}{\partial\tau}f(t''+i\tau)$ we can perform the integration over τ. We obtain,

$$\begin{aligned}&\delta\langle\pi_{\mu\nu}\rangle^l \\ &= -\frac{i}{10}H_{\mu\nu}\int d^3x'\int_{-\infty}^{t} dt' e^{\epsilon(t'-t)}\int_{-\infty}^{t'} dt''(\langle\pi(x',t''+i\beta)\pi(x,t)\rangle - \langle\pi(x',t'')\pi(x,t)\rangle)\,.\end{aligned} \tag{4.13}$$

Using the KMS condition

$$\begin{aligned}\langle\pi(x',t''+i\beta)\pi(x,t)\rangle =& \mathrm{Tr}[e^{-\beta H}\pi(x',t''+i\beta)\pi(x,t)] \\ =& \mathrm{Tr}[e^{-\beta H}\pi(x,t)\pi(x',t'')] = \langle\pi(x,t)\pi(x',t'')\rangle\end{aligned} \tag{4.14}$$

we obtain,

$$\delta\langle\pi_{\mu\nu}\rangle^l = \frac{H_{\mu\nu}}{10}\int d^3x'\int_{-\infty}^{t} dt' e^{\epsilon(t'-t)}\int_{-\infty}^{t'} dt'' D_R(x,t;x',t'') \tag{4.15}$$

where

$$D_R(x,t;x',t'') = -i\theta(t-t'')[\pi(x,t),\pi(x',t'')] \tag{4.16}$$

We extract the shear viscosity using the definition (2.3). We obtain,

$$\eta^{(1)} = \frac{1}{10}\int d^3x'\int_{-\infty}^{0} dt' e^{\epsilon(t'-t)}\int_{-\infty}^{t'} dt'' D_R(0;x',t'') \tag{4.17}$$

We can rewrite this result in a more useful form. We rewrite the integrals inserting theta functions: $\int_{-\infty}^{0} dt' = \theta(-t')\int_{-\infty}^{\infty} dt'$ and use the integral representation for the theta function:

$$\theta(x) = \int \frac{d\omega}{2\pi i}\frac{e^{i\omega x}}{\omega - i\epsilon}. \tag{4.18}$$

Inserting $D_R(0;x',t'') = \int dQ\, e^{iQX'} D_R(Q)$ with $X' = (x',t'')$ we obtain,

$$\eta^{(1)} = -\frac{i}{10}\frac{d}{dq_0}[\lim_{\vec{q}\to 0} D_R(Q)]|_{q_0=0} \tag{4.19}$$

Since it is the imaginary part of the two-point function that is odd in q_0 we have,

$$\eta^{(1)} = \frac{1}{10}\frac{d}{dq_0}\mathrm{Im}[\lim_{\vec{q}\to 0} D_R(Q)]|_{q_0=0} \tag{4.20}$$

This is the well known Kubo formula which expresses the shear viscosity in terms of a retarded two-point green function that can be calculated using equilibrium quantum field theory.

2. Quadratic Response

Now we consider corrections to the linear response approximation. We calculate the quadratic shear viscous response from the terms in (4.6) that are quadratic in the interaction. We show that the result can be written as a retarded three-point correlator. We obtain,

$$\delta\langle\pi_{\mu\nu}(x,t)\rangle^q = \int d^3x' \int d^3x'' \int_{-\infty}^{t} dt' e^{\epsilon(t'-t)} \int_{-\infty}^{t} dt'' e^{\epsilon(t''-t)}$$
$$(\pi_{\mu\nu}(x,t), \pi_{\alpha\beta}(x',t'), \pi_{\rho\sigma}(x'',t''))\partial^\alpha F^\beta(x',t')\partial^\rho F^\sigma(x'',t'')\,. \tag{4.21}$$

We use (2.2) and rotational invariance to write,

$$\pi_{\mu\nu}(x,t)\pi_{\alpha\beta}(x',t')\pi_{\rho\sigma}(x'',t'')\partial^\alpha u^\beta \partial^\lambda u^\tau = \frac{3}{35}H^{T2}_{\mu\nu}(\pi_{\alpha\beta}(x,t)\pi^{\beta\lambda}(x',t')\pi_\lambda{}^\alpha(x'',t''))$$
$$:= \frac{3}{35}H^{T2}_{\mu\nu}(\pi(x,t)\pi(x',t')\pi(x'',t'')) \tag{4.22}$$

Writing out the correlation function we obtain,

$$\delta\langle\pi_{\mu\nu}(x,t)\rangle^q = \frac{3}{35}H^{T2}_{\mu\nu} \int d^3x' \int d^3x'' \int_{-\infty}^{t} dt' e^{\epsilon(t'-t)} \int_{-\infty}^{t} dt'' e^{\epsilon(t''-t)}$$
$$[\int_0^\beta d\tau \int_0^\tau d\lambda \langle \pi(x',t'+i\tau)\pi(x'',t''+i\lambda)\pi(x,t)\rangle$$
$$-\int_0^\beta d\tau \int_0^\beta d\tau' \langle\langle \pi(x',t'+i\tau)\rangle \pi(x'',t''+i\tau')\pi(x,t)\rangle$$
$$-\int_0^\beta d\tau \int_0^\tau d\lambda \langle\langle \pi(x',t'+i\tau)\pi(x'',t''+i\lambda)\rangle \pi(x,t)\rangle$$
$$+\int_0^\beta d\tau \int_0^\beta d\tau' \langle\langle \pi(x',t'+i\tau)\rangle\langle \pi(x'',t''+i\tau')\rangle \pi(x,t)\rangle\ ] \tag{4.23}$$

Assuming that correlations vanish when the time approaches minus infinity we can rewrite (4.23) as

$$\delta\langle\pi_{\mu\nu}(x,t)\rangle^q = \frac{3}{35}H^{T2}_{\mu\nu} \int d^3x' \int d^3x'' \int_{-\infty}^{t} dt' e^{\epsilon(t'-t)} \int_{-\infty}^{t} dt'' e^{\epsilon(t''-t)} \int_{-\infty}^{t'} ds' \int_{-\infty}^{t''} ds''$$
$$\frac{d}{ds'}\frac{d}{ds''}[\int_0^\beta d\tau \int_0^\tau d\lambda \langle \pi(x',s'+i\tau)\pi(x'',s''+i\lambda)\pi(x,t)\rangle$$
$$-\int_0^\beta d\tau \int_0^\beta d\tau' \langle\langle \pi(x',s'+i\tau)\rangle \pi(x'',s''+i\tau')\pi(x,t)\rangle$$
$$-\int_0^\beta d\tau \int_0^\tau d\lambda \langle\langle \pi(x',s'+i\tau)\pi(x'',s''+i\lambda)\rangle \pi(x,t)\rangle$$
$$+\int_0^\beta d\tau \int_0^\beta d\tau' \langle\langle \pi(x',s'+i\tau)\rangle\langle \pi(x'',s''+i\tau')\rangle \pi(x,t)\rangle\ ] \tag{4.24}$$

Carrying out the integration over λ, τ and τ' and using the fact that we have symmetry under interchange of (x',t') and (x'',t'') we obtain,

$$\delta\langle\pi_{\mu\nu}(x,t)\rangle^q = \frac{3}{70}H^{T2}_{\mu\nu} \int d^3x' \int d^3x'' \int_{-\infty}^{t} dt' e^{\epsilon(t'-t)} \int_{-\infty}^{t} dt'' e^{\epsilon(t''-t)} \int_{-\infty}^{t'} ds' \int_{-\infty}^{t''} ds''$$
$$\frac{1}{2}([[\pi(x,t),\pi(x',s')],\pi(x'',s'')] + [[\pi(x,t),\pi(x'',s'')],\pi(x',s')]) \tag{4.25}$$

We define

$$G(x,t;x',s';x'',s'') = \frac{1}{2}([[\pi(x,t),\pi(x',s')],\pi(x'',s'')] + [[\pi(x,t),\pi(x'',s'')],\pi(x',s')]) \quad (4.26)$$

and write

$$\delta\langle\pi_{\mu\nu}(x,t)\rangle^q$$
$$= \frac{3}{70}H^{T2}_{\mu\nu}\int d^3x' \int d^3x'' \int_{-\infty}^{t} dt' e^{\epsilon(t'-t)} \int_{-\infty}^{t} dt'' e^{\epsilon(t''-t)} \int_{-\infty}^{t'} ds' \int_{-\infty}^{t''} ds'' G(x,t;x',s';x'',s'')$$

Using (2.3) we extract,

$$\eta^{(2)} = \frac{3}{70}\int d^3x' \int_{-\infty}^{0} dt' e^{\epsilon t'} \int_{-\infty}^{0} dt'' e^{\epsilon t''} \int_{-\infty}^{t'} ds' \int_{-\infty}^{t''} ds'' G(0;x',s';x'',s'') \quad (4.27)$$

As we did previously, we can rewrite this result in a neater form. We rewrite the integrals inserting theta functions: $\int_{-\infty}^{0} dt' = \theta(-t')\int_{-\infty}^{\infty} dt'$ and use the integral representation for the theta function (4.18). We use $X' = (x', s')$ and $X'' = (x'', s'')$ and rewrite,

$$G(0;x',s';x'',s'') = \int dQ e^{iX'Q} \int dQ' e^{iX''Q'} G(-Q-Q',Q,Q')\,. \quad (4.28)$$

We obtain,

$$\eta^{(2)} = \frac{3}{70}\frac{d}{dq_0}\frac{d}{dq_0'}[\lim_{\vec{q}\to 0} G(-Q-Q',Q,Q')]|_{q_0=q_0'=0} \quad (4.29)$$

Using the techniques of [21] to write the three-point function in the spectral representation, it is tedious but straightforward to show that the only contribution to the result above comes from the real part of the three-point function which is retarded with respect to the first leg. In coordinate space this retarded three-point function is written: $G_{R1}(x,y,z) = \theta(t_x - t_y)\theta(t_y - t_z)[[\pi(x),\pi(y)],\pi(z)] + \theta(t_x - t_z)\theta(t_z - t_y)[[\pi(x),\pi(z)],\pi(y)]$. We obtain,

$$\eta^{(2)} = \frac{3}{70}\frac{d}{dq_0}\frac{d}{dq_0'}\mathrm{Re}\,[\lim_{\vec{q}\to 0} G_{R1}(-Q-Q',Q,Q')]|_{q_0=q_0'=0}\,. \quad (4.30)$$

This is an interesting new result. We have obtained a type of nonlinear Kubo formula that allows us to obtain the quadratic shear viscous response from a retarded three-point function using equilibrium quantum field theory.

B. Diagrammatic Expansion

We obtain a diagrammatic expansion for the viscosity coefficients given in (4.20) and (4.30). We use the closed time path formulation of finite temperature field theory, and work in the Keldysh representation. Several reviews of this technique are available in the literature [22–26]. The closed time path integration contour involves two branches, one running from minus infinity to positive infinity just above the real axis, and one running back from positive infinity to negative infinity just below the real axis. All fields can take values on either branch of the contour and thus there is a doubling of the number of degrees of freedom. It is straightforward to show that this doubling of degrees of freedom is necessary to obtain finite green functions. We discuss below the structure of correlation functions of field operators.

The two-point function or the propagator can be written as a 2×2 matrix of the form

$$D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}, \tag{4.31}$$

where D_{11} is the propagator for fields moving along C_1, D_{12} is the propagator for fields moving from C_1 to C_2, etc. The four components are given by

$$\begin{aligned} D_{11}(X-Y) &= -i\langle T(\phi(X)\phi(Y))\rangle \\ D_{12}(X-Y) &= -i\langle \phi(Y)\phi(X)\rangle \\ D_{21}(X-Y) &= -i\langle \phi(X)\phi(Y)\rangle \\ D_{22}(X-Y) &= -i\langle \tilde{T}(\phi(X)\phi(Y))\rangle \end{aligned} \tag{4.32}$$

where T is the usual time ordering operator and $\tilde{T}$ is the anti-chronological time ordering operator. Physical functions are obtained by taking appropriate combinations of the components of the propagator matrix. It is straightforward to show that the usual retarded and advanced propagators: $D_R = -i\theta(x_0 - y_0)[\phi(X), \phi(Y)]$ and $D_A = -i\theta(y_0 - x_0)[\phi(X), \phi(Y)]$ are given by the combinations,

$$\begin{aligned} D_R &= D_{11} - D_{12} \\ D_A &= D_{11} - D_{21}\,. \end{aligned} \tag{4.33}$$

The 1PI part of the two-point function, or the polarization insertion, is obtained by truncating legs. The retarded and advanced parts are given by,

$$\begin{aligned} \Pi_R &= \Pi_{11} + \Pi_{12} \\ \Pi_A &= \Pi_{11} + \Pi_{21}\,. \end{aligned} \tag{4.34}$$

The situation is similar for higher n-point functions. For example, the three-point function which is retarded with respect to the first leg is given by

$$\Gamma_{R1} = \Gamma_{111} + \Gamma_{112} + \Gamma_{121} + \Gamma_{122}\,. \tag{4.35}$$

The other three-point vertices that we will need are:

$$\begin{aligned} \Gamma_{R2} &= \Gamma_{111} + \Gamma_{112} + \Gamma_{211} + \Gamma_{212} \\ \Gamma_{R3} &= \Gamma_{111} + \Gamma_{121} + \Gamma_{211} + \Gamma_{221} \\ \Gamma_{F} &= \Gamma_{111} + \Gamma_{121} + \Gamma_{212} + \Gamma_{222} \end{aligned}$$

The four-point function which is retarded with respect to the first leg is given by,

$$M_{R1} = M_{1111} + M_{1112} + M_{1121} + M_{1211} + M_{1122} + M_{1212} + M_{1221} + M_{1222}\,. \tag{4.36}$$

The other four-point vertices that we will need are:

$$
\begin{aligned}
M_{R4} &= M_{1111} + M_{1121} + M_{1211} + M_{2111} + M_{1221} + M_{2121} + M_{2211} + M_{2221} \\
M_F &= M_{1111} + M_{1121} + M_{1211} + M_{1221} + M_{2112} + M_{2212} + M_{2122} + M_{2222}\,.
\end{aligned}
\tag{4.37}
$$

There are two relations that we can use to simplify expressions involving these vertices. The first is a consequence of the fact that real time green functions are related to each other through the KMS conditions. These identities are a consequence of symmetries generated through cyclic permutations of field operators. The identity that we will need is [21],

$$
\Gamma_F + N_1\Gamma_{R3} + N_3\Gamma_{R1} = (N_1 + N_3)\Gamma^*_{R2} \tag{4.38}
$$

where each vertex carries arguments $\Gamma(P_1, P_2, P_3)$ and we write $N(P_1) := N_1$ etc. The second simplification occurs because of the fact that the equations we obtain contain a specific combination of four-point vertices which we define as,

$$
\bar{M}_F = M_F + N_1 M_{R4} + N_4 M_{R1} \tag{4.39}
$$

where each vertex carries arguments $M(P_1, P_2, P_3, P_4)$. It has been shown that this particular combination of four-point vertices has special properties in a number of other contexts [27].

In addition, bare vertices that carry a Keldysh index '2' have an extra factor of minus one. This factor is accounted for by inserting a factor of τ for each vertex where τ is the two component vector given by,

$$
\tau = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
$$

We want to obtain a perturbative expansion for the correlation functions of composite operators $D_R(X, Y)$ and $G_{R1}(X, Y, Z)$ which appear in (4.20) and (4.30). We obtain expressions for these correlation functions in terms of the vertices Γ_{ij} and M_{ijlm} which are defined as the vertices obtained by truncating external legs from the following connected vertices:

$$
\begin{aligned}
\Gamma^C_{ij} &= \langle T_c \pi_{ij}(X)\phi(Y)\phi(Z)\rangle \\
M^C_{ijlm} &= \langle T_c \pi_{ij}(X)\pi_{lm}(Y)\phi(Z)\phi(W)\rangle
\end{aligned}
\tag{4.40}
$$

where

$$
\pi_{ij}(X) = \partial_i\phi(X)\partial_j\phi(X) - \frac{1}{3}\delta_{ij}(\partial_m\phi(X))(\partial_m\phi(X)) \tag{4.41}
$$

and T_c is the operator that time orders along the closed time path contour. These definitions allow us to write the two- and three-point correlation functions as integrals of the form,

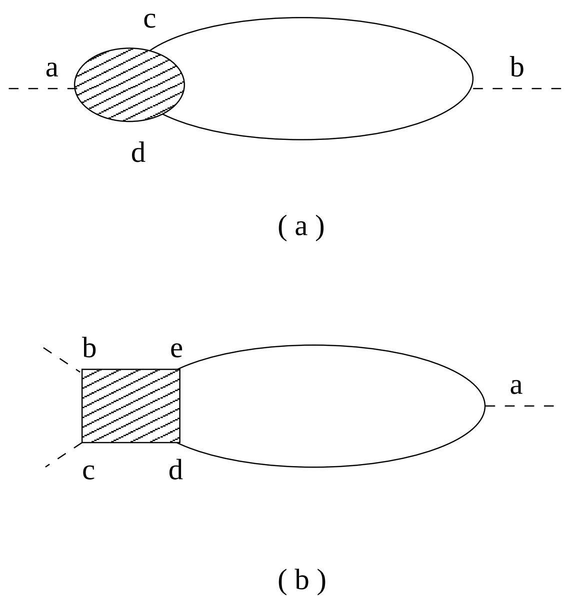

Fig. [1]: (a) Two-point function for shear viscosity from linear response; (b) Three-point function for quadratic shear viscous response. The dashed external line represents the composite operator π_{ij}. The square box is the four-point function M and the round blob is the three-point vertex Γ.

$$D_{ab}(Q) = 2i \int dP\, \Gamma^{ij}_{cad}(P, Q, -P - Q) iD_{bc}(P) iD_{db}(P + Q) I_{ji}(p)\tau_b \tag{4.42}$$
$$G_{abc}(-Q - Q', Q, Q') = 8 \int dK\, M^{ijjk}_{ebcd} \tau_a iD_{ae}(K) iD_{da}(K + Q + Q') I_{ki}(k)$$

where the indices $\{a, b, c, d, e\}$ are Keldysh indices and take values $\{1, 2\}$. These expressions are shown diagrammatically in Fig. [1]. We perform the sum over Keldysh indices using the Mathematica program described in [27]. We obtain,

$$D_R(Q) = i \int dK (N_{k+q} - N_k)\Gamma^{ij}_{R2}(K, Q, -K - Q) D_A(K) D_R(K + Q) I_{ji}(k) \tag{4.43}$$
$$\begin{aligned} &G_{R1}(-Q - Q', Q, Q') \\ &\quad = -4 \int dK (\bar{M}_F)_{ikkj}(K, Q, Q', -K - Q - Q') D_R(K) D_A(K + Q + Q') I_{ji}(k) \end{aligned} \tag{4.44}$$

To rewrite these expressions in a simpler form we use rotational invariance to write,

$$M_{ijlm} := \hat{I}_{ij}\hat{I}_{lm} M\,; \qquad \Gamma_{ij} := \hat{I}_{ij}\Gamma \tag{4.45}$$

Using an obvious notation we write the pairs of propagators $D_A(P)D_R(P + Q)$ and $D_R(K)D_A(K+Q+Q')$ as $a_p r_{p+q}$ and $r_k a_{k+q+q'}$. When $\{q_0\,,\ q_0'\} \to 0$ the dominant contribution to the integral from these pairs of propagators is produced by what is called the pinch effect: the contour is "pinched" between the poles of the two propagators which gives rise to a factor in the denominator that is proportional to the imaginary part of the propagators.

We regulate the pinching singularity with the imaginary part of the hard thermal loop self energy and obtain [15],

$$r_k a_{k+q} \to -\frac{\rho_k}{2\mathrm{Im}\Sigma_\mathrm{k}}\,; \qquad \rho_k = i(r_k - a_k) \tag{4.46}$$

where Σ is the retarded part of the hard thermal loop self energy.

Now we expand in q_0 and q_0'. In (4.43) we keep terms proportional to q_0 since these terms are the only ones that contribute to (4.20); in (4.44) we keep terms proportional to $q_0 q_0'$ since these terms are the only ones that contribute to (4.30). In each term there are products of thermal factors of the form $N_x - N_{x+q}$ and $N_x - N_{x+q'}$ where x is some combination of $\{k,\ p,\ r\}$. The expansion of these thermal factors is straightforward:

$$N_x - N_{x+q} = 2q_0\beta n_x(1+n_x) + \cdots \tag{4.47}$$

Consider the behaviour of the vertices when $\{q_0,\ q_0'\} \to 0$. Using (4.43) and (4.46) in (4.20) it is easy to see that only the real part of Γ_{R2} contributes. Similarly, using (4.44) and (4.46) in (4.30) it is clear that we need only the real part of $\bar{M}_F$. First we consider (4.43). Because of the explicit factor of $(N_{k+q} - N_k)$ we can set Q to zero in the vertex. The expansion of (4.44) in Q and Q' is more difficult. One can show that the terms in the expansion that contain gradients acting on the vertices M_F, M_{R1} and M_{R4} do not contribute to the result and that we can make the replacement:

$$\mathrm{Re}\bar{M}_F(K,Q,Q',-K-Q-Q') \to (N_k - N_{k+q+q'})\mathrm{Re}M_{R1}(K,0,0,-K) \tag{4.48}$$

From now on, to simplify the notation, we define $\Gamma(K,0,-K) := \Gamma(K)$ and $M(K,0,0,-K) := M(K)$.

Using these results to simplify (4.43) and (4.44) and substituting into (4.20) and (4.30) we obtain,

$$\eta^{(1)} = \frac{\beta}{15}\int dK\, k^2\, \rho_k n_k(1+n_k)\left[\frac{\mathrm{Re}\Gamma_\mathrm{R2}(\mathrm{K})}{\mathrm{Im}\Sigma_\mathrm{k}}\right] \tag{4.49}$$

$$\eta^{(2)} = -\frac{2\beta^2}{105}\int dK\, k^2\, \rho_k n_k(1+n_k)N_k\left[\frac{\mathrm{Re}\mathrm{M}_\mathrm{R1}(\mathrm{K})}{\mathrm{Im}\Sigma_\mathrm{k}}\right] \tag{4.50}$$

Comparing with (3.11) and (3.12) we see that the results are identical if we identify

$$B(\underline{k}) = \frac{\mathrm{Re}\Gamma_\mathrm{R2}(\underline{\mathrm{k}})}{\mathrm{Im}\Sigma_\mathrm{k}} \tag{4.51}$$

$$C(\underline{k}) = -\frac{\mathrm{Re}\mathrm{M}_\mathrm{R1}(\underline{\mathrm{k}})}{\mathrm{Im}\Sigma_\mathrm{k}} \tag{4.52}$$

with the momentum K on the shifted mass shell: $\delta(K^2 - m_{th}^2)$ where $m_{th}^2 = m^2 + \mathrm{Re}\Sigma_K$.

1. The Ladder Resummation

It has been known for some time that the set of diagrams which give the dominate contributions to the vertex $\Gamma_{ij}(P,Q,-P-Q)$ are the ladder diagrams. These diagrams contribute to the viscosity to the same order in perturbation theory as the bare one loop graph and thus need to be resummed [10]. This effect occurs for the following reason. It appears that the ladder graphs are suppressed relative to the one loop graph by extra powers of the coupling which come from the extra vertex factors that one obtains when one adds rungs (vertical lines). However, these extra factors of the coupling are compensated for by a kinematical factor. This factor arises through the pinch effect which is described above. The addition of an additional rung in a ladder graph always produces an extra pair of propagators of the form $a_k r_{k+q}$ or $r_k a_{k+q}$. Products of this form contribute a factor which produces an enhancement. This factor occurs when the contour is "pinched" between the poles of the two propagators, which gives rise to a contribution in the denominator that is proportional to the imaginary part of the inverse propagators.

In order to include ladder diagrams we obtain the vertex $\tilde{\Gamma}_{ij}(P,Q,-P-Q)$ as the solution to the integral equation shown in Fig. [2].

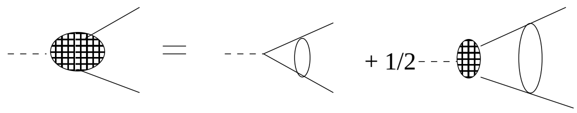

Fig. [2]: Integral equation for the ladder resummation. The blob represents the vertex $\tilde{\Gamma}$.

$$\begin{aligned}
&\tilde{\Gamma}^{lm}_{abc}(K,Q,-K-Q)\\
&= \int dP\, dR\, \tau_3^b \tau_3^a \tau_3^c I_{lm}(p) D_{ca}(R) D_{ac}(K+R-P) D_{bc}(P+Q) D_{ab}(P) \qquad (4.53)\\
&+\frac{1}{2}\int dP\, dR\, \tilde{\Gamma}^{lm}_{dbe}(P,Q,-P-Q) D_{ad}(P) D_{ec}(P+Q) D_{ac}(R+K-P)\tau_3^c D_{ca}(R)\tau_3^a
\end{aligned}$$

We perform the sums over the Keldysh indices using the Mathematica program in [27] and simplify the result by taking Q to zero, keeping only the pinching terms, and using (4.46). The vertex function which includes the tree vertex is obtained by shifting: $\Gamma_{ij} = I_{ij} + \frac{1}{2}\tilde{\Gamma}_{ij}$. (The factor of 1/2 is a symmetry factor). We obtain,

$$\begin{aligned}
\Gamma^{lm}_{R2}(K) = k_m k_l - \frac{1}{3}\delta_{ml}k^2 - \frac{\lambda^2}{4}\int dP\, dR\, dP'\\
\cdot(2\pi)^4\delta^4(P+P'-R-K)\rho_p\rho_r\rho_{p'}\frac{\Gamma^{lm}_{R2}(P)}{\mathrm{Im}\Sigma_R(P)}(1+n_p)(1+n_{p'})n_r/(1+n_k) \qquad (4.54)
\end{aligned}$$

Note that this integral equation is decoupled: the only three-point vertex that appears is Γ^{lm}_{R2}. To simplify this expression further we use (4.45), (4.51) and the fact that $\Gamma_{R2}(P)$ is pure real, and symmetrize the integral on the right hand side over the integration variables $\{P,\, P',\, R\}$. We multiply and divide the left hand side by $\mathrm{Im}\Sigma_\mathrm{k}$ and replace this expression in the numerator by the HTL result [10,28],

$$\mathrm{Im}\Sigma_k = -\frac{\lambda^2}{12}\left(\frac{1}{1+n_k}\right)\int dP\, dR\, dP'\,(2\pi)^4\delta^4(P+P'-R-K)$$
$$\rho_p\rho_r\rho_{p'}(1+n_p)(1+n_{p'})n_r \tag{4.55}$$

Rearranging we obtain [15],

$$I(k,k)_{lm} = \frac{\lambda^2}{12(1+n_k)}\int dP\, dR\, dP'\ (2\pi)^4\delta(P+P'-R-K)\rho_p\rho_r\rho_{p'} \tag{4.56}$$
$$[B_{lm}(P)+B_{lm}(P')-B_{lm}(K)-B_{lm}(R)](1+n_p)(1+n_{p'})n_r$$

where we have used (3.7), (4.45) and (4.51). When the delta functions are used to do the frequency integrals, this equation has exactly the same form as the equation obtained from the linearized Boltzmann equation (3.19) with a shifted mass shell describing effective thermal excitations. Comparing (3.11) and (3.19) with (4.49) and (4.56) we conclude that calculating shear viscosity using effective transport theory by keeping only first order terms in the gradient expansion is equivalent to using the Kubo formula obtained from linear response theory, with a three-point vertex obtained by resumming ladder graphs. The contributions to the viscosity are shown in Fig. [3].

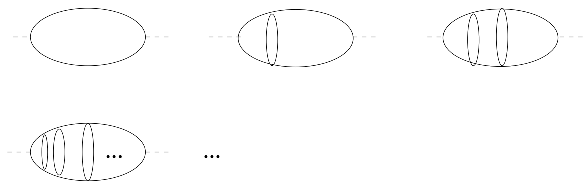

Fig. [3]: Some of the ladder diagrams that contribute to shear viscosity.

2. The Extended-Ladder Resummation

In this section we consider an integral equation for the vertex M_{ijlm} which resums an infinite set of graphs that includes ladder graphs and some other contributions which we will call extended ladder graphs. We will show that this integral equation has exactly the same form as the integral equation (3.22) obtained by expanding the Boltzmann equation to second order. We consider the integral equation shown in Fig. [4].

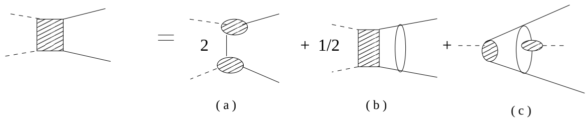

Fig. [4]: Integral equation for an extended-ladder resummation.

Following (4.39) and (4.44) we calculate contributions to M_F, M_{R1}, M_{R4} for each diagram. We introduce some notation to simplify the equations. For four-point vertices we list the first three momenta only and for three-point vertices we list the first two only. In both cases the last momentum is the one that is determined by the conservation of energy. For example: $M(K,Q,Q',-K-Q-Q') := M(K,Q,Q')$ and $\Gamma(K,Q,-K-Q) := \Gamma(K,Q)$. In addition we write (as before) $M(K,0,0,-K) := M(K)$ and $\Gamma(K,0,-K) := \Gamma(K)$. From Fig. [4a] we have,

$$\begin{aligned}
(M_F^{ijlm})^{4a}(K,Q,Q') &= 2i[r_{k+q}\Gamma_{R1}^{ij}(K,Q)\Gamma_{F2}^{lm}(K+Q,Q') \\
&\quad + a_{k+q}\Gamma_{F2}^{ij}(K,Q)\Gamma_{R3}^{lm}(K+Q,Q') + f_{k+q}\Gamma_{R1}^{ij}(K,Q)\Gamma_{R3}^{lm}(K+Q,Q')] \\
(M_{R1}^{ijlm})^{4a}(K,Q,Q') &= 2ir_{k+q}\Gamma_{R1}^{ij}(K,Q)\Gamma_{R1}^{lm}(K+Q,Q') \\
(M_{R4}^{ijlm})^{4a}(K,Q,Q') &= 2ia_{k+q}\Gamma_{R3}^{ij}(K,Q)\Gamma_{R3}^{lm}(K+Q,Q')
\end{aligned} \tag{4.57}$$

We can rewrite this result by substituting in the expanded form of the vertex $\Gamma_{ij} = I_{ij} + \frac{1}{2}\tilde{\Gamma}_{ij}$ with $\tilde{\Gamma}_{ij}$ obtained from the integral equation that corresponds to Fig. [2]. We use,

$$\begin{aligned}
&\Gamma_{R1}^{ij}(K,Q) \\
&\quad = I^{ij}(k) - \frac{i\lambda^2}{8}\int dP\,dR\; r_p a_{p+q}(N'-N_r)a_r r'\,[\Gamma_{F2}^{ij}(P,Q) - N_{p+q}\Gamma_{R1}^{ij}(P,Q) + N_p\Gamma_{R3}^{ij}(P,Q)] \\
&\Gamma_{R3}^{ij}(K,Q) \\
&\quad = I^{ij}(k) - \frac{i\lambda^2}{8}\int dP\,dR\; r_p a_{p+q}(N_r-N')r_r a'\,[\Gamma_{F2}^{ij}(P,Q) - N_{p+q}\Gamma_{R1}^{ij}(P,Q) + N_p\Gamma_{R3}^{ij}(P,Q)] \\
&\Gamma_{F2}^{ij}(K,Q) \\
&\quad = -\frac{i\lambda^2}{8}\int dP\,dR\; r_p a_{p+q}(1-N'N_r)\rho_r\rho'\,[\Gamma_{F2}^{ij}(P,Q) - N_{p+q}\Gamma_{R1}^{ij}(P,Q) + N_p\Gamma_{R3}^{ij}(P,Q)]
\end{aligned}$$

It is easy to show that the contribution from Fig. [4a] is equivalent to the contributions from the four diagrams in Fig. [5].

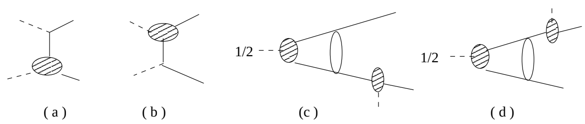

Fig. [5]: The four-point vertices that correspond to the diagram in to Fig. [4a].

From Figs. [5a,5b] we obtain,

$$(M_F^{ijlm})^{5a}(K,Q,Q') = iI_{ij}(k)[a_{k+q}\Gamma_{F2}^{lm}(K,Q) + f_{k+q}\Gamma_{R1}^{lm}(K,Q)] \tag{4.58}$$

$$(M_{R1}^{ijlm})^{5a}(K,Q,Q') = ir_{k+q}I_{ij}(k)\Gamma_{R1}^{lm}(K,Q) \tag{4.59}$$

$$(M_{R4}^{ijlm})^{5a}(K,Q,Q') = ia_{k+q}I_{ij}(k)\Gamma_{R3}^{lm}(K,Q) \tag{4.60}$$

$$(M_F^{ijlm})^{5b}(K,Q,Q') = iI_{ij}(k)[r_{k+q}\Gamma^{lm}_{F2}(K+Q,Q') + f_{k+q}\Gamma^{lm}_{R3}(K,Q)] \tag{4.61}$$
$$(M_{R1}^{ijlm})^{5b}(K,Q,Q') = ir_{k+q}I_{ij}(k)\Gamma^{lm}_{R1}(K+Q,Q') \tag{4.62}$$
$$(M_{R4}^{ijlm})^{5b}(K,Q,Q') = ia_{k+q}I_{ij}(k)\Gamma^{lm}_{R3}(K+Q,Q') \tag{4.63}$$

From Figs. [5c,5d] we obtain,

$$\begin{aligned}
&(M_F^{ijlm})^{5c}(K,Q,Q') \\
&\quad = \frac{-i\lambda^2}{8}\int dP\,dR\,a_{p+q+q'}r_p \\
&\quad [\Gamma^{lm}_{F2}(P+Q,Q') - N_{p+q+q'}\Gamma^{lm}_{R1}(P+Q,Q') + N_{p+q}\Gamma^{lm}_{R3}(P+Q,Q')] \\
&\quad \cdot [\Gamma^{ij}_{F2}(K,Q)(f_r a_{p'} + r_r f_{p'})a_{k+q} \\
&\quad + \Gamma^{ij}_{R1}(K,Q)(f_{k+q}(f_r a_{p'} + r_r f_{p'}) + r_{k+q}(f_{p'}f_r + a_{p'}r_r + r_{p'}a_r))]
\end{aligned} \tag{4.64}$$

$$\begin{aligned}
&(M_{R1}^{ijlm})^{5c}(K,Q,Q') \\
&\quad = \frac{-i\lambda^2}{8}\int dP\,dR\,a_{p+q+q'}r_p \\
&\quad [\Gamma^{lm}_{F2}(P+Q,Q') - N_{p+q+q'}\Gamma^{lm}_{R1}(P+Q,Q') + N_{p+q}\Gamma^{lm}_{R3}(P+Q,Q')] \\
&\quad \cdot \Gamma^{ij}_{R1}(K,Q)r_{k+q}(f_{p'}a_r + r_{p'}f_r)
\end{aligned} \tag{4.65}$$

$$\begin{aligned}
&(M_{R4}^{ijlm})^{5c}(K,Q,Q') \\
&\quad = \frac{-i\lambda^2}{8}\int dP\,dR\,a_{p+q+q'}r_p \\
&\quad [\Gamma^{lm}_{F2}(P+Q,Q') - N_{p+q+q'}\Gamma^{lm}_{R1}(P+Q,Q') + N_{p+q}\Gamma^{lm}_{R3}(P+Q,Q')] \\
&\quad \cdot \Gamma^{ij}_{R3}(K,Q)a_{k+q}(f_{p'}r_r + a_{p'}f_r).
\end{aligned} \tag{4.66}$$

$$\begin{aligned}
&(M_F^{ijlm})^{5d}(K,Q,Q') \\
&\quad = \frac{-i\lambda^2}{8}\int dP\,dR\,a_{p+q}r_p[\Gamma^{lm}_{F2}(P,Q) - N_{p+q}\Gamma^{lm}_{R1}(P,Q) + N_p\Gamma^{lm}_{R3}(P,Q)] \\
&\quad [\Gamma^{ij}_{F2}(K+Q,Q')r_{k+q}(a_r f' + f_r r') \\
&\quad + \Gamma^{ij}_{R3}(K+Q,Q')(f_{k+q}(a_r f' + f_r r') + a_{k+q}(f_r f' + r_r a' + a_r r'))]
\end{aligned} \tag{4.67}$$

$$\begin{aligned}
&(M_{R1}^{ijlm})^{5d}(K,Q,Q') \\
&\quad = \frac{-i\lambda^2}{8}\int dP\,dR\,r_p a_{p+q}[\Gamma^{lm}_{F2}(P,Q) - N_{p+q}\Gamma^{lm}_{R1}(P,Q) + N_p\Gamma^{lm}_{R3}(P,Q)] \\
&\quad \cdot \Gamma^{ij}_{R1}(K+Q,Q')r_{k+q}(a_r f' + f_r r')
\end{aligned} \tag{4.68}$$

$$\begin{aligned}
&(M_{R4}^{ijlm})^{5d}(K,Q,Q') \\
&\quad = \frac{-i\lambda^2}{8}\int dP\,dR\,r_p a_{p+q}[\Gamma^{lm}_{F2}(P,Q) - N_{p+q}\Gamma^{lm}_{R1}(P,Q) + N_p\Gamma^{lm}_{R3}(P,Q)] \\
&\quad \cdot \Gamma^{ij}_{R3}(K+Q,Q')a_{k+q}(a' f_r + f' r_r)
\end{aligned} \tag{4.69}$$

The contribution from Fig. [4b] is

$$(M_F^{ijlm})^{4b}(K,Q,Q') = -\frac{\lambda^2}{8}\int dP\,dR\,a_{p+q+q'}r_p\bar{M}_F^{ijlm}(P,Q,Q')[r_r a_{p'} + a_r r_{p'} + f_r f_{p'}] \quad (4.70)$$

$$(M_{R1}^{ijlm})^{4b}(K,Q,Q') = -\frac{\lambda^2}{8}\int dP\,dR\,a_{p+q+q'}r_p\bar{M}_F^{ijlm}(P,Q,Q')[a_r f_{p'} + f_r r_{p'}] \quad (4.71)$$

$$(M_{R4}^{ijlm})^{4b}(K,Q,Q') = -\frac{\lambda^2}{8}\int dP\,dR\,a_{p+q+q'}r_p\bar{M}_F^{ijlm}(P,Q,Q')[r_r f_{p'} + f_r a_{p'}] \quad (4.72)$$

where $P' = R + K - P$. Fig. [4c] gives,

$$\begin{aligned}(M_F^{ijlm})^{4c}(K,Q,Q') &= \frac{-i\lambda^2}{4}\int dP\,dR\,(N_p - N_{p+q})(N_{r-q'} - N_r) \quad (4.73)\\ &\Gamma_{R2}^{ij*}(P,Q)\Gamma_{R2}^{lm*}(R-Q',Q')f_{p'}r_p a_{r+q} a_r r_{r-q'}\\ (M_{R1}^{ijlm})^{4c}(K,Q,Q') &= \frac{-i\lambda^2}{4}\int dP\,dR\,(N_p - N_{p+q})(N_{r-q'} - N_r)\Gamma_{R2}^{ij*}(P,Q) \quad (4.74)\\ &\Gamma_{R2}^{lm*}(R-Q',Q')r_{r'}r_p a_{r+q} a_r r_{r-q'}\\ (M_{R4}^{ijlm})^{4c}(K,Q,Q') &= \frac{-i\lambda^2}{4}\int dP\,dR\,(N_p - N_{p+q})(N_{r-q'} - N_r)\Gamma_{R2}^{ij*}(P,Q) \quad (4.75)\\ &\Gamma_{R2}^{lm*}(R-Q',Q')a_{p'}r_p a_{r+q} a_r r_{r-q'}\end{aligned}$$

To combine these expressions we use (4.38) and (4.39). We keep only the pinching contributions and use (4.46) to regulate. As discussed previously, we expand in q_0 and q_0' and keep the term proportional to $q_0 q_0'$, since that is the only term that will contribute to the quadratic shear viscous response coefficient. As before, one can show that the terms in the expansion that contain gradients acting on the vertices M_F, M_{R1} and M_{R4} do not contribute to the result and that we can use (4.48). We simplify further by using (4.47). We also make repeated use of the set of identities below which hold for momenta which satisfy $P_1 + P_2 = P_3 + P_4$:

$$\begin{aligned}&n_1 n_2(1+n_3)(1+n_4) = (1+n_1)(1+n_2)n_3 n_4\\ &n_3 n_2 + n_3 + n_1(n_3 - n_2) = (1+n_1)(1+n_2)n_3/(1+n_4) \quad (4.76)\\ &n_1(1+n_1)n_3(1+n_3)(N_2 - N_4) = (1+n_1)(1+n_2)n_3 n_4(N_3 - N_1)\,.\end{aligned}$$

Finally, since K is the momentum for an external leg we take the on shell piece:

$$r_k \to \frac{i}{\mathrm{Im}\Sigma_k} \quad (4.77)$$

We obtain:

$$\begin{aligned}&N_k n_k(1+n_k)M_{R1}^{ijlm}(K)\\ &\quad = -n_k(1+n_k)N_k I_{ij}\frac{\Gamma_{R2}^{lm}(K)}{\mathrm{Im}\Sigma_k} + \frac{\lambda^2}{4}\int dP\,dR\,(1+n_p)(1+n_{p'})n_r n_k \rho_p\rho_p'\rho_r\\ &\left[-\frac{N_p M_{R1}^{ijlm}(P)}{\mathrm{Im}\Sigma_p} + \frac{1}{2}\frac{\Gamma_{R2}^{ij}(P)}{\mathrm{Im}\Sigma_p}\frac{\Gamma_{R2}^{lm}(R)}{\mathrm{Im}\Sigma_r}(N_r - N_p) + \frac{1}{2}\frac{\Gamma_{R2}^{ij}(P)}{\mathrm{Im}\Sigma_p}\frac{\Gamma_{R2}^{lm}(K)}{\mathrm{Im}\Sigma_k}(N_p - N_k)\right] \quad (4.78)\end{aligned}$$

We introduce the symmetric notation: $P := P_1$; $P' := P_2$; $R := P_3$ and rewrite the equation above after symmetrizing on the integration variables. Rearranging we obtain,

$$
\begin{aligned}
&n_k(1+n_k)N_k I_{ij}\frac{\Gamma^{lm}_{R2}(K)}{\mathrm{Im}\Sigma_k}\\
&\quad=\frac{\lambda^2}{12}\int\int dP_1\,dP_2\,dP_3(2\pi)^4\delta^4(P_1+P_2-P_3-K)(1+n_1)(1+n_2)n_3n_k\rho_1\rho_2\rho_3\\
&\quad[\frac{N_{p_3}M^{ijlm}_{R1}(P_3)}{\mathrm{Im}\Sigma_{p_3}}+\frac{N_kM^{ijlm}_{R1}(K)}{\mathrm{Im}\Sigma_k}-\frac{N_{p_1}M^{ijlm}_{R1}(P_1)}{\mathrm{Im}\Sigma_{p_1}}-\frac{N_{p_2}M^{ijlm}_{R1}(P_2)}{\mathrm{Im}\Sigma_{p_2}}\\
&\quad+\frac{1}{2}\{(N_1+N_2)\frac{\Gamma^{ij}_{R2}(P_1)}{\mathrm{Im}\Sigma_{p_1}}\frac{\Gamma^{lm}_{R2}(P_2)}{\mathrm{Im}\Sigma_{p_2}}-(N_k+N_3)\frac{\Gamma^{ij}_{R2}(K)}{\mathrm{Im}\Sigma_k}\frac{\Gamma^{lm}_{R2}(P_3)}{\mathrm{Im}\Sigma_{p_3}}\\
&\quad+(N_3-N_1)\frac{\Gamma^{ij}_{R2}(P_1)}{\mathrm{Im}\Sigma_{p_1}}\frac{\Gamma^{lm}_{R2}(P_3)}{\mathrm{Im}\Sigma_{p_3}}+(N_k-N_1)\frac{\Gamma^{ij}_{R2}(P_1)}{\mathrm{Im}\Sigma_{p_1}}\frac{\Gamma^{lm}_{R2}(K)}{\mathrm{Im}\Sigma_k}\\
&\quad+(N_3-N_2)\frac{\Gamma^{ij}_{R2}(P_3)}{\mathrm{Im}\Sigma_{p_3}}\frac{\Gamma^{lm}_{R2}(P_2)}{\mathrm{Im}\Sigma_{p_2}}+(N_k-N_2)\frac{\Gamma^{ij}_{R2}(K)}{\mathrm{Im}\Sigma_k}\frac{\Gamma^{lm}_{R2}(P_2)}{\mathrm{Im}\Sigma_{p_2}}\}]
\end{aligned}
\tag{4.79}
$$

Note that once again we have obtained an integral equation that is decoupled: it only involves M_{R1} and Γ_{R2}. With Γ_{R2} determined by the integral equation (4.56), Equation (4.79) can be solved to obtain M_{R2}. From the diagrams in Figs. [4,5] we see that the solutions to the integral equation will contain contributions of the form shown in Fig. [6]. Finally, by using (3.7), (4.45), (4.51) and (4.52) and comparing (4.50) and (4.79) with (3.12) and (3.22) we see that calculating the quadratic shear viscous response using transport theory describing effective thermal excitations and keeping terms that are quadratic in the gradient of the four-velocity field in the expansion of the Boltzmann equation, is equivalent to calculating the same response coefficient from quantum field theory at finite temperature using the next-to-linear response Kubo formula with a vertex given by a specific integral equation. This integral equation shows that the complete set of diagrams that need to be resummed includes the standard ladder graphs, and an additional set of extended ladder graphs. Some of the diagrams that contribute to the viscosity are shown in Fig. [7].

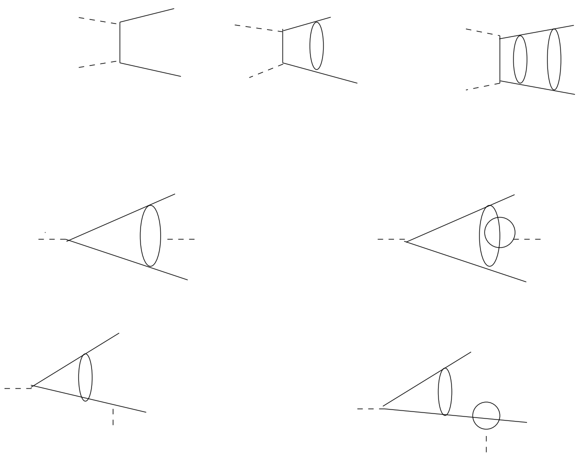

Fig. [6]: Some of the ladder and extended-ladder diagrams that contribute to the four-point vertex.

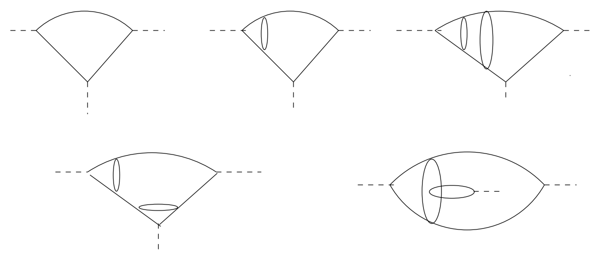

Fig. [7]: Some of the ladder and extended ladder diagrams that contribute to quadratic shear viscous response.

V. CONCLUSIONS

We have studied nonlinear response using two different methods. The first method uses standard transport theory. We start from a local equilibrium form for the distribution function and perform a gradient expansion. We calculate the quadratic shear viscous response coefficient by expanding the Boltzmann equation and obtaining a hierarchy of equations that can be solved consistently. The second technique uses response theory. We work with a perturbative Hamiltonian that is linear in the gradient of the four-velocity field and study quadratic response. We generalize the Kubo formula for linear response and obtain an expression that allows us to calculate quadratic shear viscous response from the retarded three-point green function of the viscous shear stress tensor. The transport theory calculation involves the use of the Boltzmann equation which is itself obtained from some more fundamental theory. The response theory calculation uses the well known methods of equilibrium finite temperature quantum field theory and is, in this sense, more fundamental. However, the response theory calculation is complicated by the need to resum infinite sets of diagrams at finite temperature.

At leading order, it is well known that a correct calculation of the linear response coefficient involves the resummation of ladder graphs. Beyond leading order in response theory it is difficult even to identify which diagrams need to be resummed. We have identified precisely which diagrams need to be resummed by studying the connection between the transport theory calculation and the response theory calculation. We have shown that calculating the quadratic shear viscous response coefficient using transport theory by keeping terms that are quadratic in the gradient of the four-velocity field in the expansion of the Boltzmann equation, is equivalent to calculating the same response coefficient from quantum field theory at finite temperature using the next-to-linear response Kubo formula with a vertex given by a specific integral equation. This integral equation shows that the complete set of diagrams that need to be resummed includes the standard ladder graphs, and an additional set of extended ladder graphs.

There are several directions for future work. It has been shown that the Boltzmann equation can be derived from the Kadanoff-Baym equations by using a gradient expansion and keeping only linear terms [29]. The connection between this result and the work discussed in this paper can probably be understood by studying the dual roles of the gradient expansion and the quasiparticle approximation. In addition, it would be interesting to generalize this work to gauge field theories.

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